

ON THE DIRICHLET SPACE FOR FINITELY CONNECTED REGIONS

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ABSTRACT. This paper is devoted to the study of the Dirichlet space $\text{Dir}(G)$ for finitely connected regions G ; we are particularly interested in the algebra of bounded multiplication operators on this space. Results in different directions are obtained. One direction deals with the structure of closed subspaces invariant under all bounded multiplication operators. In particular, we show that each such subspace contains a bounded function. For regions with circular boundaries we prove that a finite codimensional closed subspace invariant under multiplication by z must be invariant under all bounded multiplication operators, and furthermore it is of the form $p \text{Dir}(G)$, where p is a polynomial with all its roots lying in G . Another direction is to study cyclic and noncyclic vectors for the algebra of all bounded multiplication operators. Typical results are: if $f \in \text{Dir}(G)$ and f is bounded away from zero then f is cyclic; on the other hand, if the zero set of the radial limit function of f on the boundary has positive logarithmic capacity, then f is not cyclic. Also, some other sufficient conditions for a function to be cyclic are given. Lastly, we study transitive operator algebras containing all bounded multiplication operators; we prove that they are dense in the algebra of all bounded operators in the strong operator topology.

1. INTRODUCTION

Let G be a domain in the complex plane \mathbb{C} . The Bergman space $B(G)$ is the Hilbert space of functions f analytic in G such that $|f|^2$ is integrable with respect to the area measure dA on G ; the norm in this space is given by

$$\|f\|_{B(G)}^2 = \int_G |f|^2 dA.$$

The Dirichlet space $\text{Dir}(G)$ for the domain G is the Hilbert space of functions f analytic in G whose derivative f' lies in $B(G)$. Fixing a point w in G , we define a norm for $\text{Dir}(G)$ by

$$\|f\|_{\text{Dir}(G)}^2 = |f(w)|^2 + \int_G |f'|^2 dA = |f(w)|^2 + \|f'\|_{B(G)}^2.$$

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A function f analytic in G is in $\text{Dir}(G)$ if and only if the area (counting multiplicity) of $f(G)$ is finite. Of course, the value of the norm depends on the fixed point w in G . However, the norms obtained by fixing different points are equivalent; see, for example, Proposition 6 of [BrSh].

In case G is the open unit disk \mathbb{D} centered at the origin, the functions f in $\text{Dir}(\mathbb{D})$ have a power series representation; $f(z) = \sum_{n=0}^{\infty} a_n z^n$. The norm of $\text{Dir}(\mathbb{D})$ can be taken as

$$\|f\|_{\text{Dir}(\mathbb{D})}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'|^2 dA = |a_0|^2 + \pi \sum_{n=1}^{\infty} n |a_n|^2.$$

If f is in $\text{Dir}(\mathbb{D})$, then $\sum |a_n|^2$ is finite and hence f is also in the Hardy space H^2 . In particular, the radial limit function of f is defined almost everywhere on the boundary $\partial\mathbb{D}$ of \mathbb{D} . The polynomials are dense in $\text{Dir}(\mathbb{D})$; in fact the partial sums of the series for f converge to f in the norm topology. One can also check that if $f \in \text{Dir}(\mathbb{D})$ then $zf \in \text{Dir}(\mathbb{D})$.

A function ϕ analytic in G is said to be a multiplier for $\text{Dir}(G)$ if $\phi \text{Dir}(G) \subset \text{Dir}(G)$, that is ϕf is in $\text{Dir}(G)$ if f is in $\text{Dir}(G)$. By $M(\text{Dir}(G))$, we denote the algebra of all multipliers for $\text{Dir}(G)$. Every multiplier will automatically induce a linear transformation $M_\phi: \text{Dir}(G) \rightarrow \text{Dir}(G)$ defined by

$$M_\phi f = \phi f, \quad \text{for all } f \text{ in } \text{Dir}(G).$$

One can show that M_ϕ is a bounded linear operator; this follows from the closed graph theorem and the continuity of the linear functionals of evaluation at points in G . We call M_ϕ a multiplication operator. Since $\text{Dir}(G)$ contains constant functions, the multipliers are functions in $\text{Dir}(G)$. Also we know that a multiplier is necessarily bounded in G (see [DuRS, Lemma 11]). We have an inclusion relation: $M(\text{Dir}(G)) \subset \text{Dir}(G) \cap H^\infty(G)$, where $H^\infty(G)$ denotes the algebra of all bounded analytic functions in G . In case $\text{Dir}(G) \subset B(G)$, every function analytic in the closure of G , denoted by G^- , is a multiplier for $\text{Dir}(G)$.

We call a circular domain any domain that is obtained by removing a finite number of disjoint closed subdisks (some perhaps of radius zero) from the open unit disk. If G is a (nonempty) finitely connected domain in \mathbb{C} , there exists a conformal mapping ψ of G onto some circular domain Ω . We can choose ψ so that the composition map $C_\psi: f \rightarrow f \circ \psi$ is an isometry from $\text{Dir}(\Omega)$ onto $\text{Dir}(G)$. We shall consider circular domains Ω . Without loss of generality, we assume that each removed closed subdisk has positive radius because otherwise the removed disk is just a point and it is a removable singularity for all the functions in $\text{Dir}(\Omega)$. Since $\text{Dir}(G)$, $M(\text{Dir}(G))$, $H^\infty(G)$ are all preserved by conformal maps, those results using only these concepts are valid in arbitrary finitely connected regions.

In this paper, we shall add and multiply functions defined on different domains. Without further comment, the domain of the sum or the product will be taken to be the intersection of the domains of the summands or the factors.

2. PRELIMINARIES

In this section, we set forth some preliminaries that we shall need later. We begin by looking at the Dirichlet space $\text{Dir}(G)$ for an open connected set G in \mathbb{C} . It is easy to show that in case G is bounded, $\text{Dir}(G)$ is contained in $B(G)$ if and only if z is a multiplier for $\text{Dir}(G)$. There are some other equivalent conditions; see [AxSh]. Consider the following property.

If $w \in G$, $f \in \text{Dir}(G)$ and $f(w) = 0$, then $f(z)/(z - w)$ is in $\text{Dir}(G)$.

We remark that if $\text{Dir}(G) \subset B(G)$, then $\text{Dir}(G)$ has the above property. The converse is true when G is bounded. These follow from simple computations. It is well known that if z is a point in G and $n \geq 0$ is an integer then the linear functional $\lambda_{z,n}: f \rightarrow f^{(n)}(z)$ is norm continuous on $\text{Dir}(G)$.

Lemma 2.1. *If K is a compact subset of G and $n \geq 0$ is an integer, then there exists a constant $C = C(n, K) > 0$ such that for all z in K and all functions f in $\text{Dir}(G)$, $|f^{(n)}(z)| \leq C\|f\|_{\text{Dir}(G)}$.*

Proof. Apply the principle of uniform boundedness to the family of bounded linear functionals $\lambda_{z,n}: f \rightarrow f^{(n)}(z)$, where z is in K . \square

The analogue of this result is also true for Bergman spaces.

Lemma 2.2. *If K is a compact subset of G and $n \geq 0$ is an integer, then there exists a constant $C = C(n, K) > 0$ such that for all z in K and all functions f in $B(G)$, $|f^{(n)}(z)| \leq C\|f\|_{B(G)}$.*

Fix $0 < R < 1$. From now on we shall use the following notation.

Notation. Denote $\mathcal{R} = \{z \in \mathbb{C}: R < |z|\}$ and $\mathcal{A} = \{z \in \mathbb{C}: R < |z| < 1\}$.

The following two lemmas establish some basic facts about the Dirichlet spaces and the Bergman spaces for these two regions. The proofs follow from direct calculations using Laurent series, and they are omitted.

Lemma 2.3. *Let $f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$ be a function analytic in \mathcal{R} . We have the following.*

(1) *If f is in $\text{Dir}(\mathcal{R})$, then $a_n = 0$ for $n > 0$ and*

$$\|f'\|_{B(\mathcal{R})}^2 = \pi \sum_{n=-\infty}^{-1} |n| |a_n|^2 R^{2n}.$$

(2) *If f is in $B(\mathcal{R})$, then $a_n = 0$ for $n \geq -1$ and*

$$\|f\|_{B(\mathcal{R})}^2 = \pi \sum_{n=-\infty}^{-2} |n+1|^{-1} |a_n|^2 R^{2(n+1)}.$$

(3) If f is in $\text{Dir}(\mathcal{R})$, then f/z is also in $\text{Dir}(\mathcal{R})$ and f/z^2 is in $B(\mathcal{R})$.

Lemma 2.4. Let $f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$ be a function analytic in \mathcal{A} . We have the following.

(1) If f is in $\text{Dir}(\mathcal{A})$, then

$$\|f'\|_{B(\mathcal{A})}^2 = \pi \sum_{n \neq 0} n |a_n|^2 (1 - R^{2n}).$$

(2) If f is in $\text{Dir}(\mathcal{A})$, then zf is in $\text{Dir}(\mathcal{A})$. (Equivalently, $\text{Dir}(\mathcal{A}) \subset B(\mathcal{A})$.)

(3) If f is in $B(\mathcal{A})$, then

$$\|f\|_{B(\mathcal{A})}^2 = \pi \sum_{n \neq -1} (n+1)^{-1} |a_n|^2 (1 - R^{2(n+1)}) + 2\pi |a_{-1}|^2 \log(R^{-1}).$$

Without further comment, we shall apply the results (with the corresponding modifications) in Lemmas 2.3 and 2.4 to the Dirichlet spaces for regions that can be obtained by translating, shrinking or enlarging \mathcal{A} or \mathcal{R} .

We shall use the following definition throughout this paper.

Definition. Let $g(z) = \sum_{n=-\infty}^{+\infty} b_n (z - \zeta)^n$ be a Laurent series about the point ζ . For $0 < t < 1$, we define $g_t(z) = \sum_{n=-\infty}^{+\infty} b_n t^{|n|} (z - \zeta)^n$.

In case $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is an analytic function in \mathbb{D} , then $g_t(z) = g(tz)$ is a function analytic in $t^{-1}\mathbb{D} = \{z \in \mathbb{C} : |z| < t^{-1}\}$. If $g(z) = \sum_{n=-\infty}^{-1} b_n z^n$ is a function analytic in \mathcal{R} , then $g_t(z) = g(t^{-1}z)$ is a function analytic in $\{z \in \mathbb{C} : |z| > tR\}$. Also, if $h(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$ is analytic in \mathcal{A} , we can apply the Cauchy integral formula to write h as $h = h_0 + h_1$, where $h_0(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in \mathbb{D} and $h_1(z) = \sum_{n=-\infty}^{-1} a_n z^n$ is analytic in \mathcal{R} . By our definition, we have $h_t = (h_0)_t + (h_1)_t$, and so h_t is analytic in $\{z \in \mathbb{C} : tR < |z| < t^{-1}\}$. Consequently, $h_t \in M(\text{Dir}(\mathcal{A}))$. On the other hand it is easy to see, by using Lemma 2.3, that if f is in $\text{Dir}(\mathcal{R})$ then f_t is in $\text{Dir}(\mathcal{R})$. Using the previous lemmas, one can prove that if f is a function in $\text{Dir}(G)$ where G denotes either \mathbb{D} , \mathcal{R} or \mathcal{A} , then $f_t \rightarrow f$ in the norm topology as $t \uparrow 1$. This is also true if we replace $\text{Dir}(G)$ by $B(G)$.

If $u(z) = \sum_{n=0}^{\infty} u_n z^n$ is a function analytic in \mathbb{D} , then $v(z) = u(R/z) = \sum_{n=0}^{\infty} u_n (R/z)^n$ is a function analytic in \mathcal{R} . Furthermore, $u \in \text{Dir}(\mathbb{D})$ if and only if $v \in \text{Dir}(\mathcal{R})$, and also $u \in M(\text{Dir}(\mathbb{D}))$ if and only if $v \in M(\text{Dir}(\mathcal{R}))$. In fact $\|u'\|_{B(\mathbb{D})} = \|v'\|_{B(\mathcal{R})}$.

3. SUM AND PRODUCT OF FUNCTIONS

Let G be an open connected set in \mathbb{C} . We use $H(G)$ to denote the algebra of all analytic functions in G and $H^\infty(G)$ to denote the algebra of all bounded functions in $H(G)$. In case G is unbounded, $H_0(G)$ denotes the space of all functions in $H(G)$ that vanish at ∞ .

Throughout this paper we shall use the following notation concerning a general finitely connected circular region.

Notation. Let $\Omega = \mathbb{D} \setminus (D_1^- \cup D_2^- \cup \cdots \cup D_N^-)$ where $D_i^- = \{z: |z - \zeta_i| \leq r_i\}$ ($i = 1, \dots, N$) are disjoint closed subdisks of the open unit disk \mathbb{D} . Fix a sufficiently small positive number η so that $\eta < \inf\{\text{dist}(\partial D_i, \partial D_j): 0 \leq i, j \leq N, i \neq j\}$ (where $D_0 = \mathbb{D}$). Let $\Gamma_0 = \{z: |z| = 1 - 2^{-1}\eta\}$ and $\Gamma_i = \{z: |z - \zeta_i| = r_i + 2^{-1}\eta\}$ ($1 \leq i \leq N$) be circles lying in Ω and concentric to the boundary circles of Ω . By $\text{Int}(\Gamma_i)$ and $\text{Ext}(\Gamma_i)$ we denote the bounded and unbounded components of $\mathbb{C} \setminus \Gamma_i$ respectively. We let $A_0 = \text{Ext}(\Gamma_0) \cap \Omega$ and $A_i = \text{Int}(\Gamma_i) \cap \Omega$ ($i = 1, \dots, N$). We remark that A_i ($i = 0, \dots, N$) are annuli in Ω . Finally, we denote $\Omega_i = \mathbb{C} \setminus D_i^-$ ($i = 1, \dots, N$).

An application of the Cauchy integral formula shows that if f is in $H(\Omega)$, then we can write $f = f_0 + f_1 + \cdots + f_N$, where $f_0 \in H(\mathbb{D})$ and $f_i \in H_0(\Omega_i)$. This is also true for bounded analytic functions; it is proved in [ChPS] that

$$H^\infty(\Omega) = H^\infty(\mathbb{D}) + H_0^\infty(\Omega_1) + \cdots + H_0^\infty(\Omega_N)$$

where $H_0^\infty(\Omega_i) = H^\infty(\Omega_i) \cap H_0(\Omega_i)$. The next theorem tells us a similar decomposition phenomenon for the Dirichlet spaces.

Theorem 3.1. $\text{Dir}(\Omega) = \text{Dir}(\mathbb{D}) + \text{Dir}_0(\Omega_1) + \cdots + \text{Dir}_0(\Omega_N)$, where $\text{Dir}_0(\Omega_i) = H_0(\Omega_i) \cap \text{Dir}(\Omega_i)$.

Proof. For notational simplicity, we shall only prove the result for the case $N = 2$. If h is in $\text{Dir}(\Omega)$, then $h = h_0 + h_1 + h_2$, where $h_0 \in H(\mathbb{D})$, $h_i \in H_0(\Omega_i)$, ($i = 1, 2$). We must prove that $h_0 \in \text{Dir}(\mathbb{D})$ and $h_i \in \text{Dir}(\Omega_i)$.

To show that h_0 is in $\text{Dir}(\mathbb{D})$, we prove that $|h'_0|^2$ is integrable with respect to the area measure dA on A_0 and $\mathbb{D} \setminus A_0$ separately. On A_0 , both h'_1 and h'_2 are bounded and so they are square integrable. By our assumption h' is square integrable over A_0 and so is $h'_0 = h' - h'_1 - h'_2$. On the other hand, h'_0 is bounded on $\mathbb{D} \setminus A_0$, it is square integrable.

We now show that $h_1 \in \text{Dir}(\Omega_1)$ and it will follow from the same argument that $h_2 \in \text{Dir}(\Omega_2)$. We write, for z in Ω_1 , $h_1(z) = \sum_{n=1}^{\infty} a_n(z - \zeta_1)^{-n}$. So $h'_1(z) = -(z - \zeta_1)^{-2} \sum_{n=1}^{\infty} n a_n(z - \zeta_1)^{-n+1}$. We denote $g(z) = -(z - \zeta_1)^2 h'_1(z)$. Since $\lim_{|z| \rightarrow \infty} g(z) = a_1$ as $|z| \rightarrow \infty$, g is bounded on $\Omega_1 \setminus A_1$. Let $|g(z)| < M$ on $\Omega_1 \setminus A_1$.

$$\int_{\Omega_1 \setminus A_1} |h'_1|^2 dA \leq M^2 \iint_{\Omega_1 \setminus A_1} r^{-3} dr d\theta, \quad \text{which is finite.}$$

It remains to show that h'_1 is square integrable over A_1 . This is obvious because $h'_1 = h' - h'_0 - h'_2$, both h'_0 and h'_2 are bounded on A_1 , and h' is square integrable over A_1 .

Conversely, if $g_0 \in \text{Dir}(\mathbb{D})$ and $g_i \in \text{Dir}(\Omega_i)$ ($i = 1, 2$) then g'_0, g'_1, g'_2 are all square integrable over Ω and so is $g' = g'_0 + g'_1 + g'_2$. \square

Not only is the decomposition theorem valid for the Dirichlet spaces, it is also valid for their algebras of multipliers.

Theorem 3.2. $M(\text{Dir}(\Omega)) = M(\text{Dir}(\mathbb{D})) + M_0(\text{Dir}(\Omega_1)) + \cdots + M_0(\text{Dir}(\Omega_N))$, where $M_0(\text{Dir}(\Omega_i)) = H_0(\Omega_i) \cap M(\text{Dir}(\Omega_i))$.

Proof. We shall only prove the result for the case $N = 2$. We first show that $M(\text{Dir}(\Omega))$ is contained in the direct sum

Let $\phi \in M(\text{Dir}(\Omega))$, so $\phi \in H^\infty(\Omega) \cap \text{Dir}(\Omega)$. By Theorem 3.1, ϕ has a decomposition $\phi = \phi_0 + \phi_1 + \phi_2$, where $\phi_0 \in \text{Dir}(\mathbb{D})$ and $\phi_i \in \text{Dir}(\Omega_i)$ ($i = 1, 2$). We must show that $\phi_0 \in M(\text{Dir}(\mathbb{D}))$ and $\phi_i \in M(\text{Dir}(\Omega_i))$.

Let $f_0 \in \text{Dir}(\mathbb{D})$. We want to show that $\phi_0 f_0$ is in $\text{Dir}(\mathbb{D})$. Clearly its derivative $(\phi_0 f_0)'$ is square integrable over $\text{Int}(\Gamma_0)$. To show that $(\phi_0 f_0)'$ is square integrable over A_0 , we write $(\phi_0 f_0)' = (\phi f_0)' - (\phi_1 f_0)' - (\phi_2 f_0)'$ and use the fact that $\text{Dir}(\mathbb{D})$ is contained in $B(\mathbb{D})$.

Next we want to show that $\phi_i \in M(\text{Dir}(\Omega_i))$ ($i = 1, 2$). It is enough to consider the case $i = 1$. Let $f_1 \in \text{Dir}(\Omega_1)$. Note that f_1' is square integrable over A_1 and by Lemma 2.4, f_1 is also square integrable over A_1 , and so is $(\phi_1 f_1)' = (\phi f_1)' - (\phi_0 f_1)' - (\phi_2 f_1)'$. We observe that $(\phi_1 f_1)' = \phi_1 f_1' + (z - \zeta_1)^2 \phi_1'(z - \zeta_1)^{-2} f_1$. Clearly the first term on the right-hand side is square integrable over $\text{Ext}(\Gamma_1)$. By Lemma 2.3, $(z - \zeta_1)^{-2} f_1$ is square integrable over $\text{Ext}(\Gamma_1)$. Since ϕ_1 vanishes at ∞ , the Laurent series for ϕ_1 at ζ_1 has no constant term; one can show that $(z - \zeta_1)^2 \phi_1'(z)$ is bounded on $\text{Ext}(\Gamma_1)$. Hence $(\phi_1 f_1)'$ is square integrable over $\text{Ext}(\Gamma_1)$.

The converse can be proved by a similar method. We omit the details. \square

The next proposition establishes the fact that $\text{Dir}(\Omega) \subset B(\Omega)$.

Proposition 3.3. *The function z is in $M(\text{Dir}(\Omega))$.*

Proof. Let $f \in \text{Dir}(\Omega)$. Clearly $(zf)' = zf' + f$ is square integrable over $\Omega \setminus (A_0 \cup \cdots \cup A_N)$. By Lemma 2.4, it is also square integrable over A_0, \dots, A_N . \square

The direct sum decomposition of functions f in $H(\Omega)$ allows us to define f_t ($0 < t < 1$) as an extension of our definition for Laurent series.

Definition. Let $f = f_0 + f_1 + \cdots + f_N$ be the unique decomposition of f in $H(\Omega)$, where $f_0 \in H(\mathbb{D})$ and $f_i \in H_0(\Omega_i)$. We define $f_t = (f_0)_t + (f_1)_t + \cdots + (f_N)_t$.

Of course if $f \in \text{Dir}(\Omega)$, then $f_t \rightarrow f$ in the norm as $t \uparrow 1$.

We shall use the following notation throughout the rest of this paper.

Notation. Fixing a point w_0 in Ω , we define the norms for different Dirichlet spaces:

$$\begin{aligned} \|f\|^2 &= |f(w_0)|^2 + \int_{\Omega} |f'|^2 dA && \text{for } f \text{ in } \text{Dir}(\Omega), \\ \|f\|_0^2 &= |f(w_0)|^2 + \int_{\mathbb{D}} |f'|^2 dA && \text{for } f \text{ in } \text{Dir}(\mathbb{D}), \\ \|f\|_i^2 &= |f(w_0)|^2 + \int_{\Omega_i} |f'|^2 dA && \text{for } f \text{ in } \text{Dir}(\Omega_i) \ (1 \leq i \leq N). \end{aligned}$$

Proposition 3.4. *There is a positive constant C such that if $f_0 \in \text{Dir}(\mathbb{D})$, $f_i \in \text{Dir}(\Omega_i)$ ($1 \leq i \leq N$) and $f = f_0 f_1 \cdots f_N$ is the product function, then $\|f\| \leq C \|f_0\|_0 \|f_1\|_1 \cdots \|f_N\|_N$.*

Proof. To estimate $\int_{\Omega} |f'|^2 dA$, we write $\Omega = K \cup A_0 \cup A_1 \cup \cdots \cup A_N$ where K is a compact set, and estimate the integral over each subregion. One might use the fact that the linear map of multiplication by $(z - \zeta_m)^{-2}$ is a bounded operator from $\text{Dir}(\Omega_m)$ to $B(\Omega_m)$ (see Lemma 2.3). We omit the details. \square

4. CONVERGENCE PROPERTIES

The main result in this section is the following: if $\phi \in \text{Dir}(\Omega) \cap H^\infty(\Omega)$, $f \in \text{Dir}(\Omega)$ and $\phi f \in \text{Dir}(\Omega)$, then $\|\phi_t f - \phi f\| \rightarrow 0$ as $t \uparrow 1$. This result will be used repeatedly in the later sections. One corollary is that the operators of multiplication by rational functions with no poles in Ω^- are dense in all multiplication operators in the strong operator topology (abbreviated SOT).

Lemma 4.1. *For $f \in \text{Dir}(\mathcal{A})$, $z \in \mathcal{A}$ and $0 < t < 1$, we have*

$$|f(z) - f_t(z)|^2 \leq \frac{1}{\pi(1 - R^2)} \left(\log \frac{1 - t|z|}{1 - |z|} + \log \frac{|z| - tR}{|z| - R} \right) \|f'\|_{B(\mathcal{A})}^2.$$

Proof. Let $f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n \in \text{Dir}(\mathcal{A})$. By the Cauchy-Schwarz inequality and Lemma 2.4, we see that

$$\begin{aligned} |f(z) - f_t(z)|^2 &\leq \left(\sum_{n \neq 0} |a_n|^2 |n| |1 - R^{2n}| \right) \\ &\quad \times \left(\sum_{n \neq 0} |n|^{-1} |1 - R^{2n}|^{-1} (1 - t^{|n|})^2 |z|^{2n} \right) \\ &\leq \pi^{-1} \|f'\|_{B(\mathcal{A})}^2 \left(\sum_{n \neq 0} |n|^{-1} |1 - R^{2n}|^{-1} (1 - t^{|n|})^2 |z|^{2n} \right). \end{aligned}$$

We first estimate the terms with positive powers.

$$\begin{aligned} \sum_{n>0} n^{-1} (1 - R^{2n})^{-1} (1 - t^n)^2 |z|^{2n} &\leq (1 - R^2)^{-1} \sum_{n>0} n^{-1} (1 - t^n) |z|^n \\ &= (1 - R^2)^{-1} \{ \log(1 - |z|t) - \log(1 - |z|) \}. \end{aligned}$$

Similarly we can show that

$$\begin{aligned} \sum_{n<0} |n|^{-1} |1 - R^{2n}|^{-1} (1 - t^{|n|})^2 |z|^{2n} \\ \leq (1 - R^2)^{-1} \{ \log(|z| - Rt) - \log(|z| - R) \}. \quad \square \end{aligned}$$

Lemma 4.2. *For $f \in \text{Dir}(\mathcal{A})$, $R^{1/2} < t < 1$ and $Rt^{-1} \leq |z| \leq t$, we have*

$$|f(z) - f_t(z)|^2 \leq \pi^{-1} (1 - R^2)^{-1} (\log 2 + \log(1 + R^{-1/2})) \|f'\|_{B(\mathcal{A})}^2.$$

Proof. The maximum value of $|f(z) - f_t(z)|^2$ is attained on the boundary. \square

By Theorem 3.1 (with $\Omega = \mathcal{A}$), every function h in $\text{Dir}(\mathcal{A})$ can be written as $h = h_0 + h_1$, where $h_0 \in \text{Dir}(\mathbb{D})$ and $h_1 \in \text{Dir}_0(\mathcal{A})$. Furthermore there exist positive constants C_0 and C_1 depending on R such that

$$\|(h_0)'\|_{B(\mathbb{D})} \leq C_0 \|h'\|_{B(\mathcal{A})}, \quad \text{and} \quad \|(h_1)'\|_{B(\mathcal{A})} \leq C_1 \|h'\|_{B(\mathcal{A})}.$$

Lemma 4.3. *There exists a positive constant C depending on R such that if f, g are in $\text{Dir}(\mathcal{A})$ and $R^{1/2} < t < 1$, then*

$$\int_{\mathcal{A}} |f - f_t|^2 |(g_t)'|^2 dA < C \|f'\|_{B(\mathcal{A})}^2 \|g'\|_{B(\mathcal{A})}^2.$$

Proof. Let $f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$ and $g(z) = \sum_{n=-\infty}^{+\infty} b_n z^n$ be functions in $\text{Dir}(\mathcal{A})$ and $R^{1/2} < t < 1$ (so, $R < R/t < t < 1$).

Throughout this proof, C will denote a constant which depends only on R , however it may represent different values as the proof goes along. We can rewrite the left-hand side of the inequality in the statement of the lemma as

$$\begin{aligned} (4.1) \quad & \int_0^{2\pi} \int_R^1 |f - f_t|^2 |(g_t)'|^2 r dr d\theta \\ &= \left(\int_0^{2\pi} \int_R^{R/t} + \int_0^{2\pi} \int_{R/t}^t + \int_0^{2\pi} \int_t^1 \right) |f - f_t|^2 |(g_t)'|^2 r dr d\theta. \end{aligned}$$

It follows from Lemma 4.2 that the second summand above is bounded by an expression in the form of the right-hand side of the inequality. Next we show that the same is true for the third summand.

Let $g_0(z) = \sum_{n=0}^{\infty} b_n z^n$ and $g_1(z) = \sum_{n=-\infty}^{-1} b_n z^n$, and for $R < r < 1$, let

$$a(r) = \int_0^{2\pi} |(g_0)'(re^{i\theta})|^2 d\theta = 2\pi \sum_{n=1}^{\infty} |b_n|^2 n^2 r^{2(n-1)}.$$

Then $a(r)$ is an increasing function of r and so,

$$a(r) \int_r^1 s ds \leq \int_r^1 a(s) s ds = \int_r^1 \int_0^{2\pi} |(g_0)'(se^{i\theta})|^2 s d\theta ds.$$

Thus, if $R < r < 1$, then $a(r)(1-r)(1+r)/2 \leq \|(g_0)'\|_{B(\mathcal{A})}^2$. Hence,

$$\int_0^{2\pi} |[(g_0)_t]'(re^{i\theta})|^2 d\theta \leq 2C_0^2(1-rt)^{-1} \|g'\|_{B(\mathcal{A})}^2.$$

On the other hand, by Lemma 2.2,

$$\text{Max}\{|[(g_1)_t]'(z)|^2 : t \leq |z| \leq 1\} \leq C_1^2 C \|g'\|_{B(\mathcal{A})}^2.$$

Hence,

$$\begin{aligned} & \int_0^{2\pi} |(g_t)'(re^{i\theta})|^2 d\theta \\ &= \int_0^{2\pi} |[(g_0)_t]'(re^{i\theta})|^2 d\theta + \int_0^{2\pi} |[(g_1)_t]'(re^{i\theta})|^2 d\theta \\ &\leq C(1-rt)^{-1} \|g'\|_{B(\mathcal{A})}^2. \end{aligned}$$

Together with Lemma 4.1, this implies that

$$\begin{aligned} & \int_0^{2\pi} \int_t^1 |f - f_t|^2 |(g_t)'|^2 r dr d\theta \\ & \leq C \|f'\|_{B(\mathcal{A})}^2 \|g'\|_{B(\mathcal{A})}^2 \left\{ \int_t^1 (1-rt)^{-1} \log[(1-rt)(1-r)^{-1}] dr \right. \\ & \quad \left. + \int_t^1 (1-rt)^{-1} \log[(r-tR)(r-R)^{-1}] dr \right\}. \end{aligned}$$

It remains to show that the last two integrals are bounded by a constant which depends on R only. The first one is bounded by $2 \log 2$ (see [Shie, Lemma 3]). For the second one, we can make trivial estimations.

For the first summand of the equation (4.1), we let

$$b(r) = \int_0^{2\pi} |(g_1)'(re^{i\theta})|^2 d\theta = 2\pi \sum_{n=-\infty}^{-1} |b_n|^2 n^2 r^{2(n-1)},$$

and we proceed similarly. We leave the details to the readers. \square

Lemma 4.4. Suppose $f, g \in \text{Dir}(\mathcal{A})$, then

$$\lim_{t \uparrow 1} \int_{\mathcal{A}} |f - f_t|^2 |(g_t)'|^2 dA = 0 \quad \text{as } t \uparrow 1.$$

Proof. We first note that the lemma is true if f is a polynomial in z and $1/z$. This is because $f_t \rightarrow f$ uniformly on the closure of \mathcal{A} .

Let $\varepsilon > 0$, we choose a polynomial $p = p(z, z^{-1})$ (a polynomial in z and z^{-1}) such that $\|f' - p'\|_{B(\mathcal{A})} < \varepsilon$. Let $h = f - p$. By the triangle inequality,

$$\|(f - f_t)(g_t)'\|_{B(\mathcal{A})} \leq \|(p - p_t)(g_t)'\|_{B(\mathcal{A})} + \|(h - h_t)(g_t)'\|_{B(\mathcal{A})}.$$

Now our result follows from Lemma 4.3. \square

Lemma 4.5. If $g \in \text{Dir}(\mathbb{D})$ and $h \in \text{Dir}(\mathcal{R})$, then

$$\lim_{t \uparrow 1} \|(gh)_t - g_t h_t'\|_{B(\mathcal{A})} = 0 \quad \text{as } t \uparrow 1.$$

Proof. By the triangle inequality, we have

$$\begin{aligned} & \|[(gh)_t]' - [g_t h_t']\|_{B(\mathcal{A})} \leq \|[(gh)_t]' - [gh]'\|_{B(\mathcal{A})} \\ & \quad + \|[gh]' - [g_t h_t']\|_{B(\mathcal{A})} + \|[g_t h_t]' - [g_t h_t']\|_{B(\mathcal{A})}. \end{aligned}$$

Clearly the first summand goes to zero as $t \uparrow 1$. The third summand is bounded by $\|(g_t)'(h - h_t)\|_{B(\mathcal{A})} + \|g_t(h - h_t)'\|_{B(\mathcal{A})}$. By Lemma 4.4, $\|(g_t)'(h - h_t)\|_{B(\mathcal{A})} \rightarrow 0$ as $t \uparrow 1$. For the second term, we note that $g_t(z)$ is bounded on $R < |z| < (1+R)/2$ and apply Lemma 2.1 to $h(z) - h_t(z)$ on $(1+R)/2 < |z| < 1$.

The second summand can be shown to go to zero by a similar method. \square

If f, g are in $\text{Dir}(\mathcal{A})$, then we can write $f = f_0 + f_1$, $g = g_0 + g_1$ where f_0, g_0 are in $\text{Dir}(\mathbb{D})$ and f_1, g_1 are in $\text{Dir}(\mathcal{R})$. Notice that

$$(4.2) \quad (fg)_t - f_t g_t = (f_0 g_1)_t + (f_1 g_0)_t - (f_0)_t (g_1)_t - (f_1)_t (g_0)_t.$$

The following lemma will be generalized in Theorem 4.7. However it is a useful tool in proving that.

Lemma 4.6. Suppose $g \in H^\infty(\mathcal{A}) \cap \text{Dir}(\mathcal{A})$, $f \in \text{Dir}(\mathcal{A})$ and $gf \in \text{Dir}(\mathcal{A})$. Then

$$\|[g_t f - gf]'\|_{B(\mathcal{A})} \rightarrow 0 \text{ as } t \uparrow 1.$$

Proof. Applying the triangle inequality, we have

$$\begin{aligned} \|[g_t f - gf]'\|_{B(\mathcal{A})} &\leq \|[g_t(f - f_t)]'\|_{B(\mathcal{A})} \\ &\quad + \|[g_t f_t - (gf)_t]'\|_{B(\mathcal{A})} + \|[(gf)_t - gf]'\|_{B(\mathcal{A})}. \end{aligned}$$

It is clear that the first and third summand go to zero as $t \uparrow 1$. For the second summand, we can use (4.2) and Lemma 4.5. \square

Suppose X is a Banach space and T is a bounded linear operator on X . A closed subspace M of X is said to be an invariant subspace for T if $TM \subseteq M$. For $g \in \text{Dir}(\Omega)$, we let $[g]$ denote the smallest closed subspace containing g and invariant under $M_z, M_{1/(z-\zeta_1)}, \dots, M_{1/(z-\zeta_N)}$. If ψ is a function analytic in Ω^- then it is a uniform limit of polynomials in $z, (z - \zeta_1)^{-1}, \dots, (z - \zeta_N)^{-1}$. So $\psi f \in [f]$ if $f \in \text{Dir}(\Omega)$.

Theorem 4.7. If $\phi \in H^\infty(\Omega) \cap \text{Dir}(\Omega)$, $f \in \text{Dir}(\Omega)$ and $\phi f \in \text{Dir}(\Omega)$, then $\|\phi_t f - \phi f\| \rightarrow 0$ as $t \uparrow 1$, and $\phi f \in [f]$.

Proof. We must show that

$$\int_{\Omega} |(\phi_t f - \phi f)'|^2 dA \rightarrow 0 \text{ as } t \uparrow 1.$$

We write $\Omega = K \cup A_0 \cup \dots \cup A_N$ where K is a compact set. It follows from Lemma 2.1 that the integral over K goes to zero as $t \uparrow 1$. For the integrals over the annuli, one might use the decomposition $\phi = \phi_0 + \phi_1 + \dots + \phi_N$ where $\phi_0 \in H^\infty(\mathbb{D}) \cap \text{Dir}(\mathbb{D})$ and $\phi_i \in H_0^\infty(\Omega_i) \cap \text{Dir}(\Omega_i)$ ($1 \leq i \leq N$). Now the theorem is an easy consequence of Lemma 4.6. \square

Theorem 4.8. $\{M_\phi: \phi \in M(\text{Dir}(\Omega))\} = \text{SOT-closure of } \{M_r: r \in \mathcal{G}\}$, where \mathcal{G} is the set of all rational functions with poles only at ζ_1, \dots, ζ_N .

Proof. Let $\{r_\alpha\} \subset \mathcal{G}$ be a net such that $M_{r_\alpha} \rightarrow T$ (SOT) where T is a bounded operator on $\text{Dir}(\Omega)$. Let $\psi = T1$, where we use 1 to denote the constant function of value 1. So for all $z \in \Omega$, $|r_\alpha(z) - \psi(z)| \rightarrow 0$ and similarly if $f \in \text{Dir}(\Omega)$, $|r_\alpha(z)f(z) - (Tf)(z)| \rightarrow 0$. Hence $(Tf)(z) = (\psi f)(z)$ in Ω .

The converse is a corollary of last theorem. \square

5. CYCLIC VECTORS

A vector in a Banach space is said to be a cyclic vector for an algebra of (bounded linear) operators on the space if the only closed subspace containing the vector and invariant under all operators in the algebra is the whole space.

If the algebra is generated by a single operator, then the vector is said to be a cyclic vector for that operator. In what follows, we say that a function $g \in \text{Dir}(\mathcal{A})$ or $g \in \text{Dir}(\Omega)$ is a cyclic vector when it is a cyclic vector for the corresponding algebra of all multiplication operators. Also by \mathcal{P}_k ($k \geq 1$) we denote the set of all complex polynomials of k variables, and by \mathcal{P} we denote $\{p(z, (z - \zeta_1)^{-1}, \dots, (z - \zeta_N)^{-1}) : p \text{ is in } \mathcal{P}_{N+1}\}$.

It is clear that $g \in \text{Dir}(\Omega)$ is cyclic if and only if $[g] = \text{Dir}(\Omega)$ and that \mathcal{P} is dense in $\text{Dir}(\Omega)$. We remark the following.

Remarks. For $p \in \mathcal{P}$ and $f, g \in \text{Dir}(\Omega)$,

- (i) $p[f] \subset [f]$.
- (ii) If $g \in [f]$, then $[g] \subset [f]$.
- (iii) If $g \in [f]$ and $[g] = \text{Dir}(\Omega)$, then $[f] = \text{Dir}(\Omega)$.
- (iv) $[f] = \text{Dir}(\Omega)$ if and only if $1 \in [f]$.
- (v) $[f] = \text{Dir}(\Omega)$ if and only if there exists a sequence $\{p_k\}$ in \mathcal{P} such that $p_k f \rightarrow 1$ in norm.
- (vi) $[f] = \text{Dir}(\Omega)$ if and only if there exists a sequence $\{p_k\}$ in \mathcal{P} such that $p_k f \rightarrow 1$ weakly.
- (vii) $[f] = \text{Dir}(\Omega)$ if and only if there exists a sequence $\{p_k\}$ in \mathcal{P} such that $p_k(z)f(z) \rightarrow 1$ for all z in Ω , and $\|p_k f\| \leq \text{constant}$.

We remark that (i) follows directly from the fact that $p \in M(\text{Dir}(\Omega))$. It is clear that (i) implies (ii) and (ii) implies (iii). Since \mathcal{P} is dense in $\text{Dir}(\Omega)$, $[1] = \text{Dir}(\Omega)$ and now (iv), (v) and (vi) follows from (iii). Lastly (vii) is a straightforward application of a corollary in [BrSh, p. 272].

If a function f in $\text{Dir}(\Omega)$ vanishes at a point z_0 in Ω , then $[f] \neq \text{Dir}(\Omega)$. On the other hand, if $|f|$ is bounded below in some sense then f has a chance to be cyclic. The analogue of next theorem for the case of the unit disk is contained in [BrSh]. With slight modifications, the proofs there also work for our situation and they are omitted.

Theorem 5.1. *Let $f \in \text{Dir}(\Omega)$, $g \in H^\infty(\Omega) \cap \text{Dir}(\Omega)$, and $\phi \in M(\text{Dir}(\Omega))$. We have the following.*

- (1) ϕf is cyclic if and only if both ϕ and f are cyclic.
- (2) If $|g(z)| \leq |f(z)|$ in Ω and g^2 is cyclic then f is cyclic.
- (3) If $|\phi(z)| \leq |f(z)|$ in Ω and ϕ is cyclic then f is cyclic.
- (4) If g^2 is cyclic then g is cyclic.
- (5) If $f \in H^\infty(\Omega) \cap \text{Dir}(\Omega)$ and fg is cyclic then f and g are both cyclic.

Corollary 5.2. *If $f \in \text{Dir}(\Omega)$ and there exists a positive constant C such that $|f(z)| > C$ in Ω , then f is cyclic.*

Proof. Take $\phi(z) = C$ in the result (3) of Theorem 5.1. \square

We know that $\text{Dir}(\mathbb{D})$ and $\text{Dir}(\Omega)$ are both closed under multiplication by z . In order to indicate the underlying Hilbert space that a multiplication operator is acting, we write $(M_z, \text{Dir}(\mathbb{D}))$ and $(M_z, \text{Dir}(\Omega))$ to distinguish the two

operators of multiplication by z acting on $\text{Dir}(\mathbb{D})$ and $\text{Dir}(\Omega)$ respectively. Similar notation will be used for other multiplication operators.

We have shown that if $f_0 \in \text{Dir}(\mathbb{D})$, $f_i \in \text{Dir}(\Omega_i)$, then the product function $f = f_0 f_1 \cdots f_N$ is in $\text{Dir}(\Omega)$. The proposition that we are about to prove establishes the fact that the product of cyclic vectors is again cyclic.

Proposition 5.3. *Suppose $f_0 \in \text{Dir}(\mathbb{D})$ and $f_i \in \text{Dir}(\Omega_i)$ ($1 \leq i \leq N$). If f_0 is cyclic for $(M_z, \text{Dir}(\mathbb{D}))$ and each f_i is cyclic for $(M_{1/(z-\zeta_i)}, \text{Dir}(\Omega_i))$ ($1 \leq i \leq N$), then the product function $f = f_0 f_1 \cdots f_N$ is cyclic.*

Proof. Let p_0 be a polynomial in z and p_i be a polynomial in $(z - \zeta_i)^{-1}$ ($1 \leq i \leq N$). So $p_0 \in M(\text{Dir}(\mathbb{D}))$, and $P_i \in M(\text{Dir}(\Omega_i))$ ($1 \leq i \leq N$). We let η_i be the norm of the operator of multiplication by M_{p_i} on the corresponding Dirichlet space ($0 \leq i \leq N$). By repeated applications of Proposition 3.4 (for different finitely connected regions), there exist positive constants C_1, C_2, \dots, C_N independent of p_i, f_i such that

$$\begin{aligned} \|pf - 1\| &\leq C_N \eta_{N-1} \cdots \eta_0 \|f_{N-1}\|_{N-1} \cdots \|f_0\|_0 \|p_N f_N - 1\| \\ &\quad + C_{N-1} \eta_{N-2} \cdots \eta_0 \|f_{N-2}\|_{N-2} \cdots \|f_0\|_0 \|p_{N-1} f_{N-1} - 1\| + \cdots + \|p_0 f_0 - 1\|. \end{aligned}$$

We can choose p_0, \dots, p_N so that the right-hand side of the inequality above is as small as we like. Our result follows from Remark (iv). \square

We have known, from Corollary 5.2, that if g is analytic and nonvanishing on Ω^- , then g is cyclic. Up to now we do not know what happens if g does vanish at some points on the boundary. We remark that $u(z) \in \text{Dir}(\mathbb{D})$ is cyclic for $(M_z, \text{Dir}(\mathbb{D}))$ if and only if $v(z) = u(R/z)$ is cyclic for $(M_{1/z}, \text{Dir}(\mathcal{R}))$.

Lemma 5.4. *If $p = p(z)$ is a polynomial in z and p has no roots in Ω , then p is cyclic.*

Proof. Factorize $p = p_0 p_1 \cdots p_N$ where p_i is a polynomial in z , p_0 has all its roots lying in $\mathbb{C} \setminus \mathbb{D}$ and p_i has all its roots lying in D_i^- . We know that p_0 is cyclic for $(M_z, \text{Dir}(\mathbb{D}))$ (see [BrSh, p. 289]).

Fix $N \geq i \geq 1$. We may assume p_i is monic and let $p_i(z) = (z - a_1) \cdots (z - a_k)$ where $|a_j - \zeta_i| \leq r_i$ ($1 \leq j \leq k$). Let $b_j = a_j - \zeta_i$ ($1 \leq j \leq k$).

The polynomial $s(z) = (r_i - b_1 z) \cdots (r_i - b_k z)$ has no roots lying in \mathbb{D} , and hence it is cyclic for $(M_z, \text{Dir}(\mathbb{D}))$. Since $s(r_i/(z - \zeta_i)) = r_i^k M_{1/(z - \zeta_i)}^k p_i$ and so, p_i is cyclic for $(M_{1/(z - \zeta_i)}, \text{Dir}(\Omega_i))$. Our lemma is now an easy consequence of the last proposition. \square

Theorem 5.5. *If f is analytic on Ω^- and it has no zeros in Ω , then f is cyclic.*

Proof. We have $f = pg$, where p is a polynomial with no zeros in Ω and $|g|$ is bounded away from zero in Ω . The polynomial p is a multiplier and it is also cyclic. The result follows from Theorem 5.1 and Corollary 5.2. \square

For f in $\text{Dir}(\Omega)$, the following limits exist almost everywhere:

$$\lim_{r \uparrow 1} f(re^{i\theta}), \quad \lim_{r \downarrow r_k} f(\zeta_k + re^{i\theta}) \quad (1 \leq k \leq N).$$

We define the following zero sets of radial limits for a function f in $\text{Dir}(\Omega)$.

$$Z_0(f) = \{e^{i\theta} : \lim_{r \uparrow 1} f(re^{i\theta}) = 0 \text{ as } r \uparrow 1\},$$

$$Z_k(f) = \{e^{i\theta} : \lim_{r \downarrow r_k} f(\zeta_k + re^{i\theta}) = 0 \text{ as } r \downarrow r_k\} \quad (1 \leq k \leq N).$$

We shall prove that if any one of these sets has positive logarithmic capacity, then f cannot be cyclic. This result can be viewed as a partial converse of Corollary 5.2. However, it is not known whether f is cyclic if none of these zero sets has positive logarithmic capacity. This is not clear even for the case of $\text{Dir}(\mathbb{D})$. We require some preliminaries.

Theorem 5.6 (Brown and Shields [BrSh]). *If $f \in \text{Dir}(\mathbb{D})$ and $Z(f) = \{e^{i\theta} : \lim_{r \uparrow 1} f(re^{i\theta}) = 0 \text{ as } r \uparrow 1\}$ has positive logarithmic capacity, then f is not cyclic for $(M_z, \text{Dir}(\mathbb{D}))$.*

Corollary 5.7. *If $f \in \text{Dir}(\mathcal{R})$ and $Z(f) = \{e^{i\theta} : \lim_{r \downarrow R} f(re^{i\theta}) = 0 \text{ as } r \downarrow R\}$ has positive logarithmic capacity, then f is not cyclic for $(M_{1/z}, \text{Dir}(\mathcal{R}))$.*

Before we consider the case of a general bounded finitely connected circular region, we will take a look at a simpler case for the annulus \mathcal{A} .

Lemma 5.8. *If $f \in \text{Dir}(\mathcal{A})$, and if $Z_0(f) \cup Z_1(f)$ has positive logarithmic capacity, then f is not cyclic.*

Proof. We only prove the case $Z_0(f)$ has positive logarithmic capacity. We define orthogonal projections $Q_0, Q_1 : \text{Dir}(\mathcal{A}) \rightarrow \text{Dir}(\mathcal{A})$ by $Q_0(h) = h_0$, and $Q_1(h) = h_1$ where $h = h_0 + h_1$, $h_0 \in \text{Dir}(\mathbb{D})$ and $h_1 \in \text{Dir}(\mathcal{R})$.

Let $J_n = \{e^{i\theta} \in Z_0(f) : |f_0(re^{i\theta})| \leq n, R < r < 1\}$. These are Borel sets and $Z_0(f) = \bigcup_{n=1}^{\infty} J_n$. There exists an integer M such that J_M has a positive capacity. Hence there is a compact set $F \subset J_M$ and a Borel probability measure μ supported on F , with finite energy integral. This means that

$$I(\mu) = \iint \log \frac{1}{|z - w|} d\mu(z) d\mu(w) < \infty.$$

It is shown in the proof of Theorem 5 of [BrSh] that the linear functional $\lambda : \text{Dir}(\mathbb{D}) \rightarrow \mathbb{C}$ defined by $\lambda(h) = \lim_{r \uparrow 1} \int_F h(rw) d\mu(w)$ is continuous.

Define linear functionals $\alpha, \beta : \text{Dir}(\mathcal{A}) \rightarrow \mathbb{C}$ by

$$\alpha(h) = \lim_{r \uparrow 1} \int_F (Q_0 h)(rw) d\mu(w), \quad \text{and} \quad \beta(h) = \int_F (Q_1 h)(w) d\mu(w).$$

In fact $\alpha = \lambda \circ Q_0$ and so it is continuous. One might use Lemma 2.1 to show that β is continuous. Let p be a polynomial in z and $1/z$, we have,

$$(\alpha + \beta)(pf) = \lim_{r \uparrow 1} \int_F (pf)(rw) d\mu(w).$$

Notice that $Q_0(pf) = Q_0(pf_0 + pf_1) = pf_0 + q$ where q is a polynomial in z and $1/z$. So $Q_0(pf)(rw)$ is bounded for $w \in J_M$ and $R < r < 1$. By the bounded convergence theorem, $(\alpha + \beta)(f) = 0$ and hence f is not cyclic. \square

Theorem 5.9. *If $f \in \text{Dir}(\Omega)$ and one of the sets $Z_0(f), Z_1(f), \dots, Z_N(f)$ has positive logarithmic capacity, then f is not cyclic.*

Proof. We only consider the case $Z_0(f)$ has positive logarithmic capacity. Note that the restriction of f on A_0 is a function in $\text{Dir}(A_0)$ and is not cyclic in $\text{Dir}(\mathcal{A}_0)$ by the last lemma. One can show that the constant function 1 does not lie in $[f]$ and hence f is not cyclic. \square

6. INVARIANT SUBSPACES

In what follows, by invariant subspace, we always mean a closed subspace of $\text{Dir}(\Omega)$ which is invariant under all multiplication operators.

If \mathcal{M} is an invariant subspace of $\text{Dir}(\Omega)$ and $f \in \mathcal{M}$, then $[f] \subset \mathcal{M}$. Let ξ be a fixed point in Ω and $\mathcal{M} = \{f \in \text{Dir}(\Omega) : f(\xi) = 0\}$. Being the kernel of a continuous linear functional, \mathcal{M} is closed. It is obvious that \mathcal{M} is invariant. As we shall see later, every invariant subspace of codimension 1 is of this form. We need two lemmas whose proofs are straightforward and are omitted.

Lemma 6.1. *Let \mathcal{M} be a closed finite codimensional subspace of a Banach space \mathcal{X} . Let $T : \mathcal{X} \rightarrow \mathcal{X}$ be an invertible bounded linear operator. We have the following.*

- (1) *If $T\mathcal{M} \subset \mathcal{M}$, then $T^{-1}\mathcal{M} \subset \mathcal{M}$.*
- (2) *If $T\mathcal{M} \subset \mathcal{M}$, then $T\mathcal{M} = \mathcal{M}$.*

Lemma 6.2. *If $p(z) = (z - \alpha_1) \cdots (z - \alpha_k)$ is a polynomial where $\alpha_i \in \Omega$, then*

(1) *$p \text{Dir}(\Omega) = \{f \in \text{Dir}(\Omega) : f(\alpha_1) = \cdots = f(\alpha_k) = 0\}$ where in case of zeros of p of multiplicity larger than one, we require derivatives of appropriate orders to equal zero.*

- (2) *$\dim \text{Dir}(\Omega)/p \text{Dir}(\Omega) = \deg p = k$.*

The following result is a complete characterization of all finite codimensional invariant subspaces of $\text{Dir}(\Omega)$. A similar characterization for Bergman spaces was obtained by Axler and Bourdon in [AxBo].

Theorem 6.3. *Let \mathcal{M} be a finite codimensional closed subspace of $\text{Dir}(\Omega)$. The following are equivalent.*

- (1) *$z\mathcal{M} \subseteq \mathcal{M}$.*
- (2) *$(z - a)^{-1}\mathcal{M} \subset \mathcal{M}$ for some $a \notin \Omega^-$.*
- (3) *$(z - a)^{-1}\mathcal{M} = \mathcal{M}$ for some $a \notin \Omega^-$.*
- (4) *$(z - a)^{-1}\mathcal{M} \subset \mathcal{M}$ for all $a \notin \Omega^-$.*
- (5) *$(z - a)^{-1}\mathcal{M} = \mathcal{M}$ for all $a \notin \Omega^-$.*
- (6) *$\phi\mathcal{M} \subseteq \mathcal{M}$ for all ϕ in $M(\text{Dir}(\Omega))$.*
- (7) *$\mathcal{M} = q \text{Dir}(\Omega)$ where q is a polynomial with all its roots lying in Ω .*

Proof. We need only to show (1) implies (7). The other implications are obvious. Suppose (1) is true. Define a linear operator $Y : \text{Dir}(\Omega)/\mathcal{M} \rightarrow \text{Dir}(\Omega)/\mathcal{M}$ by $Y(f + \mathcal{M}) = zf + \mathcal{M}$. There is a nonzero polynomial q with degree at most $\dim \text{Dir}(\Omega)/\mathcal{M}$ such that $q(Y) = 0$, that is $q \text{Dir}(\Omega) \subset \mathcal{M}$.

Factorize $q = q_1 q_2$ where q_1 is a polynomial whose roots lie inside Ω and q_2 is a polynomial with no roots in Ω . By Lemma 5.4, $q_2 \text{Dir}(\Omega)$ is dense in $\text{Dir}(\Omega)$. Using the fact that q_1 is a multiplier, one can verify that $q_1 \text{Dir}(\Omega) \subset (q_1 q_2 \text{Dir}(\Omega))^\perp = (q \text{Dir}(\Omega))^\perp \subset \mathcal{M}$. By Lemma 6.2, we get

$$\begin{aligned} \dim \text{Dir}(\Omega)/\mathcal{M} &\leq \dim \text{Dir}(\Omega)/q_1 \text{Dir}(\Omega) \\ &= \deg q_1 \leq \deg q \leq \dim \text{Dir}(\Omega)/\mathcal{M}. \end{aligned}$$

Hence q is a constant multiple of q_1 and $\mathcal{M} = q \text{Dir}(\Omega)$. \square

Although the image set of a function in a Dirichlet space has finite area (counting multiplicity), the function need not be bounded. However, it is a quotient of two bounded Dirichlet functions. To prove that we require a result of Nguyen Xuan Uy. Here, we denote $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, the extended plane.

Theorem 6.4 (Nguyen Xuan Uy [Nguy]). *If K is a compact subset of \mathbb{C} with positive area then there is a nonconstant function g analytic in $\mathbb{C}^* \setminus K$ such that g and g' are bounded.*

Theorem 6.5. *If G is an open connected subset of \mathbb{C} and $f \in \text{Dir}(G)$, then there exist $g, h \in H^\infty(G) \cap \text{Dir}(G)$ such that $f = g/h$.*

Proof. The area (counting multiplicity) of the image set $f(G)$ is finite. There exists a compact set K in $\mathbb{C} \setminus f(G)$ with positive area. By Theorem 6.4, there is a function p analytic in $\mathbb{C}^* \setminus K$ such that p, p' are in $H^\infty(\mathbb{C} \setminus K)$. We may assume $p(\infty) = 0$. The function $h(z) = p \circ f(z)$ is in $H^\infty(G) \cap \text{Dir}(G)$. Let $g = fh$. It remains to show that $g \in H^\infty(G) \cap \text{Dir}(G)$.

Since $p(\infty) = 0$, we write, $p(z) = a_1 z^{-1} + a_2 z^{-2} + \dots$ for large $|z|$. We let $w = f(z)$ and so $g(z) = wp(w)$ which is bounded in $\mathbb{C}^* \setminus K$. The function $wp'(w)$ is bounded in $\mathbb{C} \setminus K$ and so $g' = f'(p \circ f) + f(p' \circ f)f'$ is in $B(G)$. \square

Next we want to show that each nontrivial invariant subspace contains bounded functions, and any two nontrivial invariant subspaces have a nontrivial intersection. This result suggests the possibility of a description of the invariant subspaces analogous to that obtained by Beurling for the Hardy space H^2 in the unit disk \mathbb{D} , and generalized by several authors to finitely connected regions. However, it is shown in [BeFP] that there exists a family $\{M_\alpha : \alpha \in \mathbb{C}\}$ of closed subspaces of $B(\mathbb{D})$ invariant under multiplication by z such that $M_\alpha \cap M_\beta = \{0\}$ if $\alpha \neq \beta$.

Theorem 6.6. *If \mathcal{M} and \mathcal{N} are two nonzero invariant subspaces of $\text{Dir}(\Omega)$, then $\mathcal{M} \cap H^\infty(\Omega) \neq \{0\}$, and $\mathcal{M} \cap \mathcal{N} \neq \{0\}$.*

Proof. We let f be a nonzero function in \mathcal{M} and let $f = g/h$ where g and h are in $\text{Dir}(\Omega) \cap H^\infty(\Omega)$. By Theorem 4.7, $h_t f \rightarrow g$ as $t \uparrow 1$ and so $g \in [\mathcal{M}]$.

We let f_1, f_2 be two bounded functions in \mathcal{M} and \mathcal{N} respectively. The product $f_1 f_2 \in \text{Dir}(\Omega)$. As $t \uparrow 1$, $(f_1)_t f_2 \rightarrow f_1 f_2$. So $f_1 f_2 \in \mathcal{N}$, and by symmetry, $f_1 f_2 \in \mathcal{M}$. \square

7. TRANSITIVE OPERATOR ALGEBRAS

Let X be a Banach space and $\mathcal{L}(X)$ be the algebra of all bounded operators on X . An operator subalgebra (containing the identity) \mathcal{E} of $\mathcal{L}(X)$ is said to be a transitive algebra if there is no nontrivial closed subspace of X invariant under every operator of \mathcal{E} . Let k be a positive integer. An algebra $\mathcal{E} \subset \mathcal{L}(X)$ is said to be k -fold transitive if for any choice of elements $x_1, \dots, x_k, y_1, \dots, y_k$ in X with x_1, \dots, x_k linearly independent, there exists a sequence $\{A_j\}_1^\infty \subset \mathcal{E}$ such that for $1 \leq i \leq k$,

$$\lim A_j x_i = y_i \quad \text{as } j \rightarrow \infty.$$

In fact \mathcal{E} is transitive if and only if \mathcal{E} is 1-fold transitive. In [Arve] Arveson remarked that if \mathcal{E} is k -fold transitive for all positive integers k then \mathcal{E} is SOT dense in $\mathcal{L}(X)$. In the same paper he proved that in case X is a Hilbert space, every transitive operator algebra containing the unilateral shift must be SOT dense in $\mathcal{L}(X)$.

By $X^{(k)}$, we denote the Banach space $X \oplus \dots \oplus X$ (k copies) consisting of vectors of the form $\langle x_1, \dots, x_k \rangle$, $x_i \in X$. Let $U_i: \mathfrak{D}_i \rightarrow X$ ($i = 1, 2$) be linear transformations defined on a vector subspace \mathfrak{D}_i of X . We write $U_1 \subset U_2$ if $\Gamma(U_1) \subset \Gamma(U_2)$ where $\Gamma(U_i)$ denotes the graph of U_i . If \mathcal{E} is a subalgebra of $\mathcal{L}(X)$ and $U: \mathfrak{D} \rightarrow X$ is a linear transformation, we say that U commutes with \mathcal{E} if for every B in \mathcal{E} , $BU \subset UB$.

Theorem 7.1 (Arveson [Arve]). *Let X be a Banach space and \mathcal{E} be a transitive subalgebra of $\mathcal{L}(X)$. Then \mathcal{E} is not 2-fold transitive if and only if there exists a closed densely defined, nonscalar linear transformation that commutes with \mathcal{E} .*

Let $k \geq 2$, and suppose \mathcal{E} is k -fold transitive but not $(k+1)$ -fold transitive. Then there exist k linear transformations T_1, \dots, T_k defined on a common domain \mathfrak{D} , such that

- (1) *each T_j commutes with \mathcal{E} ,*
- (2) *no T_j is closable,*
- (3) *$\{\langle x, T_1 x, \dots, T_k x \rangle: x \in \mathfrak{D}\}$ is a closed subspace of $X^{(k+1)}$.*

We shall prove that if $\mathcal{E} \subset \mathcal{L}(\text{Dir}(\Omega))$ is a transitive operator algebra and \mathcal{E} contains all the multiplication operators then \mathcal{E} is SOT dense in $\mathcal{L}(\text{Dir}(\Omega))$. We need to show that \mathcal{E} is k -fold transitive for all $k \geq 2$. If ψ is a function defined on Ω , we define the vector subspace $\mathfrak{D}(M_\psi) = \{f \in \text{Dir}(\Omega): \psi f \in \text{Dir}(\Omega)\}$ to be the domain of the linear transformation of multiplication by ψ . In fact $\psi \in M(\text{Dir}(\Omega))$ if and only if $\mathfrak{D}(M_\psi) = \text{Dir}(\Omega)$.

Lemma 7.2. *Let \mathcal{E} be a subalgebra of $\mathcal{L}(\text{Dir}(\Omega))$ containing $\{M_\phi: \phi \in M(\text{Dir}(\Omega))\}$ and let $T_1, \dots, T_k: \mathfrak{D} \rightarrow \text{Dir}(\Omega)$ be k linear transformations defined on the vector subspace \mathfrak{D} of $\text{Dir}(\Omega)$. Suppose each T_j commutes with \mathcal{E} and $\mathfrak{M} = \{\langle f, T_1 f, \dots, T_k f \rangle: f \in \mathfrak{D}\}$ is closed in $\text{Dir}(\Omega)^{(k+1)}$. Then there exist u_i and v_i in $H^\infty(\Omega) \cap \text{Dir}(\Omega)$ such that $T_i \subset M_{\phi_i}$, where $\phi_i = u_i/v_i$ ($1 \leq i \leq k$).*

Proof. We shall show that for all f, h in \mathfrak{D} , $T_j f/f = T_j h/h$ ($1 \leq j \leq k$), and hence if we let $\phi_j = T_j f/f$ then ϕ_j is independent of f and $T_j f = \phi_j f$. By Theorem 6.5, ϕ_j is a quotient of functions in $H^\infty(\Omega) \cap \text{Dir}(\Omega)$.

$$f = \frac{f_{1,0}}{f_{2,0}}, \quad \text{and} \quad T_j f = \frac{f_{1,j}}{f_{2,j}} \quad (1 \leq j \leq k),$$

$$h = \frac{h_{1,0}}{h_{2,0}}, \quad \text{and} \quad T_j h = \frac{h_{1,j}}{h_{2,j}} \quad (1 \leq j \leq k),$$

where $f_{1,j}$, $f_{2,j}$ and $h_{1,j}$, $h_{2,j}$ are in $H^\infty(\Omega) \cap \text{Dir}(\Omega)$. We let

$$\xi = f_{2,0} f_{2,1} \cdots f_{2,k} h_{1,0} h_{2,1} \cdots h_{2,k},$$

and

$$\eta = f_{1,0} f_{2,1} \cdots f_{2,k} h_{2,0} h_{2,1} \cdots h_{2,k}.$$

Note that $\xi, \eta \in \text{Dir}(\Omega)$ and $\xi f = \eta h$. We shall show that $T_j(\xi f)/\xi f = T_j f/f$, and by symmetry, $T_j(\eta h)/\eta h = T_j h/h$, and hence $T_j f/f = T_j h/h$. The functions $\xi f, \xi T_1 f, \dots, \xi T_k f$ are in $H^\infty(\Omega) \cap \text{Dir}(\Omega)$. By hypothesis,

$$\langle \xi_t f, T_1(\xi_t f), \dots, T_k(\xi_t f) \rangle = \langle \xi_t f, \xi_t(T_1 f), \dots, \xi_t(T_k f) \rangle$$

is in \mathfrak{M} . As $t \uparrow 1$, it tends to $\langle \xi f, \xi(T_1 f), \dots, \xi(T_k f) \rangle$. Since \mathfrak{M} is closed in $\text{Dir}(\Omega)^{(k+1)}$, $\xi f \in \mathfrak{D}$ and $\xi(T_j f) = T_j(\xi f)$. \square

Theorem 7.3. *If $\mathcal{E} \subset \mathcal{L}(\text{Dir}(\Omega))$ is a transitive operator algebra and \mathcal{E} contains $M_z, M_{1/(z-\zeta_1)}, \dots, M_{1/(z-\zeta_N)}$, then \mathcal{E} is SOT dense in $\mathcal{L}(\text{Dir}(\Omega))$.*

Proof. We may assume that \mathcal{E} contains all multiplication operators by Theorem 4.8. We first show that \mathcal{E} is 2-fold transitive. Let $U: \mathfrak{D}(U) \rightarrow \text{Dir}(\Omega)$ be a closed densely defined linear transformation that commutes with \mathcal{E} . Applying Lemma 7.2 with $k = 1$, $\mathfrak{D} = \mathfrak{D}(U)$, $\mathfrak{M} = \Gamma(U)$, we conclude that $U \subset M_\phi$ for some ϕ meromorphic in Ω . We shall show that $\phi = c$.

For each w in Ω there exists k_w in $\text{Dir}(\Omega)$ such that $f(w) = (f, k_w)$ for all f in $\text{Dir}(\Omega)$ where (\cdot, \cdot) is the inner product. If $f \in \mathfrak{D}(M_\phi)$, then $(M_\phi f, k_w) = (f, \overline{\phi(w)} k_w)$. So k_w is in the domain of M_ϕ^* , and $M_\phi^* k_w = \overline{\phi(w)} k_w$. Since $U \subset M_\phi$, $M_\phi^* \subset U^*$ and $U^* k_w = M_\phi^* k_w = \overline{\phi(w)} k_w$.

For $A \in \mathcal{E}$, $A^* U^* \subset U^* A^*$ and $U^* A^* k_w = A^* U^* k_w = \overline{\phi(w)} A^* k_w$. The algebra $\mathcal{E}^* = \{A^*: A \in \mathcal{E}\}$ is transitive. Thus $\mathcal{E}^* k_w$ is dense in $\text{Dir}(\Omega)$. The adjoint operator U^* is closed and densely defined. For g in $\text{Dir}(\Omega)$ there exist $g_i = A_i^* k_w \in \mathcal{E}^* k_w$ ($i \geq 1$) such that $g_i \rightarrow g$ in $\text{Dir}(\Omega)$. We observe that

$$\langle g_i, U^* g_i \rangle = \langle g_i, U^* A_i^* k_w \rangle = \langle g_i, \overline{\phi(w)} A_i^* k_w \rangle = \langle g_i, \overline{\phi(w)} g_i \rangle,$$

which converges to $\langle g, \overline{\phi(w)} g \rangle$. But U^* has a closed graph, so $U^* g = \overline{\phi(w)} g$. We conclude that ϕ is a constant.

Next we show that \mathcal{E} is k -fold transitive for $k \geq 3$. Suppose $U_1, \dots, U_k: \mathfrak{D} \rightarrow \text{Dir}(\Omega)$ are k linear transformations defined on a common domain $\mathfrak{D} \subset \text{Dir}(\Omega)$ and each U_i commutes with \mathcal{E} . We further assume that $\mathfrak{M} = \{\langle f, U_1 f, \dots, U_k f \rangle: f \in \mathfrak{D}\}$ is closed in $\text{Dir}(\Omega)^{(k+1)}$.

By Lemma 7.2, there exist ϕ_i ($1 \leq i \leq k$) meromorphic in Ω such that $T_i \subset M_{\phi_i}$. We shall show that T_1 is closable. Let $\{f_m\}$ be a sequence in $\mathfrak{D}(M_{\phi_1})$, and f, g be two functions in $\text{Dir}(\Omega)$ satisfying $\langle f_m, \phi_1 f_m \rangle \rightarrow \langle f, g \rangle$ in $\text{Dir}(\Omega)^{(k+1)}$. Hence for each z in Ω , $g(z) = \phi_1(z)f(z)$. That is $g = M_{\phi_1}f$. \square

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